Lecture 3

An Example That Serves as a "Bridge"

(3.1) $$u_t = d u_{xx} + r u \qquad \text{on} \quad (0,1) \times (0,\infty)$$

$$u(0,t) = u(1,t) = 0 \qquad \text{on} \quad (0,\infty)$$

$$u(x,0) = f(x)$$

Separation of variables, revisited (Strauss, Wiley and Sons 2008)

Look for solutions of the form $u(x,t) = X(x)T(t)$ and then try to represent the general solution as an infinite series of such solutions

We are led to

$$\frac{T'(t)}{T(t)} = \frac{d X''(x) + r X(x)}{X(x)}$$

$$\Rightarrow \quad d X'' + r X = \sigma X$$

$$T'(t) = \sigma T$$

Boundary conditions $\Rightarrow$ $X(0) = 0 = X(1)$

So $$X'' + \frac{(r-\sigma)}{d} X = 0$$

$$X(0) = 0 = X(1)$$

$$\Rightarrow \frac{r-\sigma}{d} = n^2\pi^2 \Leftrightarrow \sigma = r - dn^2\pi^2$$

with $$X = X_n(x) = k_1 \sin(n\pi x)$$

$$\Rightarrow \quad T = T_n(t) = k_2 e^{(r-dn^2\pi^2)t}$$

So we write the general solution in the form

(3.2) $$u(x,t) = \sum_{n=1}^{\infty} b_n e^{(r-dn^2\pi^2)t} \sin(n\pi x)$$

We need the constants b_n to be such that

(3.3) $$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\Rightarrow \quad b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \qquad \text{(Fourier series)}$$

(3.2) and (3.3) involve infinite series of functions which may fail to converge for some (perhaps all) values of x and/or t, depending on b_n. We also want (3.2) to represent convergence to a function that can be differentiated once in t and twice in x so as to satisfy (3.1)

We can impose conditions on f (equivalently the b_n), but there are a number of choices for what is meant by convergence. For example, we could require

$$\int_0^1 [f(x)]^2 dx < \infty$$

$$\Rightarrow \left(\tfrac{1}{2}\right) \sum_{n=1}^{\infty} b_n^2 = \int_0^1 [f(x)]^2 dx \quad \text{(Parseval's Identity)}$$

$$\Rightarrow \lim_{m \to \infty} \int_0^1 \left[f(x) - \sum_{n=1}^{m} b_n \sin(n\pi x) \right]^2 dx = 0$$

(mean square convergence)

But $\int_0^1 [f(x)]^2 dx < \infty$ does not force $f(x)$ to be continuous and does not guarantee

$$\lim_{m\to\infty} \sum_{n=1}^{m} b_n \sin(n\pi x) = f(x)$$

for all x.

On the other hand, suppose f' is continuous, $f(0) = 0 = f(1)$

$$\int_0^1 [f'(x)]^2 dx < \infty.$$

Let $a_n = 2\int_0^1 f'(x)\cdot\cos(n\pi x)\,dx$

and consider $\sum_{n=1}^{\infty} a_n \cos(n\pi x)$.

Then $(1/2)\sum_{n=1}^{\infty} a_n^2 = \int_0^1 [f'(x)]^2 dx$ and

$$\lim_{m\to\infty} \int_0^1 \left[f'(x) - \sum_{n=1}^{m} a_n \cos(n\pi x)\right]^2 dx = 0$$

Now integration by parts $\Longrightarrow$

$$n b_n = a_n \iff b_n = (1/n)\, a_n$$

$$\Rightarrow \quad |b_n| \le \frac{1}{2}\left[\left(\tfrac{1}{n}\right)^2 + a_n^2 \right]$$

$$\Rightarrow \sum_{n=1}^{\infty} |b_n| < \infty$$

Now $|b_n \sin(n\pi x)| \le |b_n|$

$\Rightarrow \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ converges uniformly

$$\Rightarrow \lim_{m \to \infty} \sup_{x \in [0,1]} \left| f(x) - \sum_{n=1}^{m} b_n \sin(n\pi x) \right| = 0$$

(Note that $\sum_{n=1}^{\infty} |b_n| < \infty \Rightarrow |b_n| \to 0$ as $n \to \infty$

$\Rightarrow |b_n| < 1$ for $n \ge k$ for some k

$\Rightarrow |b_n|^2 < |b_n|$ for $n \ge k \Rightarrow \sum_{n=1}^{\infty} b_n^2 < \infty$)

The point of this exposition is that to make sense of (3.1) we must define a state space of initial data $f(x)$, and the way we choose a metric on the space actually determines what the space is.

Requiring $\sum_{n=1}^{\infty} b_n^2 < \infty$ imposes different conditions on the b_n that requiring $\sum_{n=1}^{\infty} |b_n| < \infty$ and hence define different subsets of the set of infinite series. So in dealing with (3.1), we must choose a metric that allows us to measure the distance between functions (or between sequences $\{b_n\}$) and the way we measure actually determines what the space is.

The choices we have illustrated are

$$d(f,g) = \left[\int_0^1 [f(x) - g(x)]^2 dx \right]^{1/2} \qquad (L^2([0,1]))$$

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)| \qquad (C[0,1])$$

(The subspace of $C([0,1])$ of functions f with $f(0) = 0 = f(1)$ will be denoted $C_0([0,1])$.)

All of these are Banach spaces which we may want to use as state spaces in our analyses of reaction-diffusion models. Which space we use may depend on what we assume about the coefficients in the equation or system.

To verify that $L^2([0,1])$ and $C_0([0,1])$ might be legitimate state spaces for (3.1), let's consider (3.2). Set

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x)$$

with $u_n(t) = b_n e^{(r - dn^2\pi^2)t}$.

For $n > \frac{1}{\pi}\left(\sqrt{\frac{r}{d}}\right)$, $|u_n(t)| \leq |b_n|$ for $t \geq 0$.

So if $\sum_{n=1}^{\infty} b_n^2 < \infty$, then $\sum_{n=1}^{\infty} |u_n^2(t)| < \infty$

for all t. So it is reasonable to use either

$L^2[0,1]$ or the space of all functions $f(x)$ on $[0,1]$

whose Fourier coefficients $\{b_n\}$ satisfy

$$\sum_{n=1}^{\infty} |b_n| < \infty$$

as state spaces.

Note also that for $n > \sqrt{r/d}/\pi$, $u_n(t) \to 0$

as $t \to \infty$. So for large t only coefficients

b_n for $n \leq \sqrt{r/d}/\pi$ matter. In effect,

the model squeezes state space down toward

the finite dimensional subspace spanned by

$\{\sin(n\pi x) \mid n \leq \sqrt{r/d}/\pi\}$

If we want (3.2) to satisfy (3.1)

we need u_t and u_{xx} to make sense. Suppose the sequence b_n is bounded (which is true if either $\sum_{n=1}^{\infty} |b_n| < \infty$ or $\sum_{n=1}^{\infty} b_n^2 < \infty$).

If we formally differentiate term by term wrt x we get

$$\sum_{n=1}^{\infty} b_n n\pi e^{(r-dn^2\pi^2)t} \cos(n\pi x)$$

which is majorized by

$$\sum_{n=1}^{\infty} B_n(t)$$

where $B_n(t) = B_0 n\pi e^{(r-dn^2\pi^2)t}$

where $|b_n| \leq B_0$.

Now $\lim_{n\to\infty} \left|\frac{B_{n+1}}{B_n}\right|$

$$= \lim_{n\to\infty} \left(\left[\frac{n+1}{n}\right] e^{(n^2-(n+1)^2)d\pi^2 t} \right) = 0$$

for any $t > 0$

Thus $\sum_{n=1}^{\infty} B_n(t)$ converges for all $t > 0$

by the ratio test.

So $\sum_{n=1}^{\infty} b_n n\pi e^{(r - dn^2\pi^2)t} \cos(n\pi x)$

converges uniformly for any given $t > 0$.

A similar calculation will establish that

(3.2) can be differentiated term by term any

number of times in t or x, for $t > 0$.

So all such series converge to continuous functions

which are the derivatives of $u(x,t)$.

The key point is that the model (3.1) has the

effect of smoothing out solutions to (3.1).

Even if $f(x)$ is discontinuous, $u(x,t)$ will

be differentiable infinitely many times in both

x and t for $t > 0$. This smoothing property is a key feature of reaction-diffusion models. It will often imply that the model maps bounded sets into compact sets. Let's see how this works in our example.

Let B_0 be a fixed constant. Let

$$F_0 = \{ f \in L^2[0,1] \mid \int_0^1 [f(x)]^2 \leq B_0^2 \}$$

Let F_t be the set of solutions $w(x,t)$ of (3.1), evaluated at time t, so that $w(x,0) \in F_0$.

Fix a value of $t > 0$. Suppose $\{w_k\}$ is a sequence in F_t so that $w_k(x,0) = f_k(x) \in F_0$. If $f_k(x) \in F_0$, we may write

$$f_k(x) = \sum_{n=1}^{\infty} b_{nk} \sin(n\pi x)$$

where $\sum_{k=1}^{\infty} b_{nk}^2 \leq B_0^2$ (Fourier series theory)

We have

$$w_k(x,t) = \sum_{n=1}^{\infty} b_{n,k} e^{(r-dn^2\pi^2)t} \sin(n\pi x)$$

$$\frac{\partial w_k}{\partial x}(x,t) = \sum_{n=1}^{\infty} b_{n,k}\, n\pi\, e^{(r-dn^2\pi^2)t} \cos(n\pi x)$$

(both series are majorized by series we know converge for $t > 0$, independent of x and k)

So $\{w_k\}$ and $\left\{\frac{\partial w_k}{\partial x}\right\}$ are bounded uniformly in k and x. Since $\left\{\frac{\partial w_k}{\partial x}\right\}$ is bounded uniformly in x and k, we have

$$|w_k(x,t) - w_k(x',t)| \leq C\,|x-x'|$$

for all $k \geq 1$ and $x, x' \in [0,1]$

Ascoli - Arzeala (Rudin 1976) $\Longrightarrow$

$\{w_k(x,t)\}$ has a subsequence $\{w_{k_i}(x,t)\}$ which converges uniformly on $[0,1]$ to a continuous function $w(x)$. Thus

$$\lim_{i\to\infty} \int_0^1 (w_{k_i}(x,t) - w(x))^2 \, dx = 0$$

So $w_{k_i}(x,t)$ converges to $w(x)$ in $L^2[0,1]$. Thus the set F_t of solutions to (3.1) evaluated at time t whose initial data belong to F_0 is compact.

Note that F_0 is not compact. The functions $f_k(x) = B_0 \sin(k\pi x)$ satisfy

$$\int_0^1 [f_k(x)]^2 dx$$

$$= B_0^2 \int_0^1 \sin^2(k\pi x) dx$$

$$= B_0^2 \int_0^1 \frac{1-\cos(2k\pi x)}{2} dx = \frac{B_0^2}{2}$$

So $\{f_k\} \in F_0$. However

$$d(f_k, f_\ell)^2 = \int_0^1 [B_{0,1} \sin(k\pi x) - B_{0,2} \sin(\ell\pi x)]^2 dx$$

$$= B_0^2 \int_0^1 [\sin^2(k\pi x) - 2\sin(k\pi x)\sin(\ell\pi x) + \sin^2(\ell\pi x)] dx$$

$$= B_0^2 \left[\tfrac{1}{2} - 0 + \tfrac{1}{2} \right] = B_0^2$$

Now back to (3.2). The solution predicts that all solutions to (3.1) will decay to 0 if $r/d < \pi^2$, while at least some solutions grow exponentially if $r/d > \pi^2$.

So the model gives a criterion for the growth rate r needed to balance the loss of individuals across the boundary if they disperse by diffusion at rate d.

But before we get too carried away (and

this is all really cool), we should check that solutions corresponding to positive initial data remain nonnegative as time evolves. Suppose $w(x,t)$ solves (3.1) on $(0,1) \times (0,T]$, is continuous on $[0,1] \times [0,T]$ with $w(x,0) \geq 0$. Let

$$v(x,t) = e^{-at} w(x,t)$$

Then

$$v_t = -av + e^{-at} w_t$$

(3.4) $$= d\, v_{xx} + (r-a) v$$

Choose a large enough so that $r - a < 0$.

Suppose $w(x,t) < 0$ for some (x,t). $v(x,t) < 0$ at the same point. So the minimum of v is negative (v is continuous on $[0,1] \times [0,T]$, so it has a minimum.)

$v \geq 0$ when $x = 0$ or $x = 1$ or $t = 0$,

so the minimum occurs at a point

$(x_0, t_0) \in (0,1) \times (0,T]$

where (3.1) and hence (3.4) are satisfied.

If $(x_0, t_0) \in (0,1) \times (0,T)$, $v_t = 0$

and $v_{xx} \geq 0 \Rightarrow (r-a)\, v(x_0, t_0) \leq 0$ $\otimes$

since $r - a < 0$ and $v(x_0, t_0) < 0$

If $(x_0, t_0) \in (0,1) \times \{T\}$, we can only

guarantee that $v_t \leq 0$. But then

we still obtain the same contradiction.

So we must have $w(x,t) \geq 0$ for

$(x,t) \in [0,1] \times [0,T]$

Such a result is an example of what is

called a maximum principle (although we applied it to a minimum).

Note that w_1 and w_2 are solutions to (3.1)) with $w_1(x,0) \geq w_2(x,0)$, $w_1 - w_2$ is a solution with $(w_1 - w_2)(x,0) \Rightarrow (w_1 - w_2)(x,t)$ $\Rightarrow w_1(x,t) \geq w_2(x,t)$, so that the model is order-preserving.